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THE VIBRATION OF A SIMPLY SUPPORTED RECTANGULAR ELASTIC PLATE DUE TO PIEZOELECTRIC ACTUATORS

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Abstract—An analysis of a simply supported rectangular elastic plate forced into bending vibrations by the application of time harmonic voltages to piezoelectric actuators attached to its bottom and top surfaces is performed by using the equations of linear elasticity. The actuators have been modeled as thin surface films and mixed edge conditions are employed to simulate simple supports.

1. INTRODUCTION

Elastic plates with piezoelectric films attached to them have been of great interest because of their use in smart structures. Most analytical analyses were based on various approximate two-dimensional plate theories [see e.g. Tauchert (1992), Tang and Xu (1995) and Mitchell and Reddy (1995)]. The method of Fourier series has been used in analyzing deformations of elastic plates when three-dimensional equations of elasticity are used (Srinivas *et al.*, 1970; Wittrick, 1987). The boundary conditions of a simply supported plate characterized by the vanishing of the deflection and bending moment at the edges translate into mixed boundary conditions. Here the Fourier series method is employed to analyse the time harmonic bending vibration of a simply supported rectangular elastic plate with piezoelectric actuators bonded to its bottom and top surfaces. The elastic plate can be laminated and made of orthotropic or isotropic materials. The piezoelectric actuators are modeled as thin films (Tiersten, 1993) as in a recent paper by Zhou and Tiersten (1994) on the cylindrical bending of an elastic plate with piezoelectric actuators. The analysis performed here can be considered as an extension of the Fourier series analysis on the cylindrical bending vibrations of a laminated elastic plate under piezoelectric actuators (Yang *et al.*, 1994).

2. FORMULATION OF THE PROBLEM

2.1. *Governing equations*

We consider an N-layer laminated elastic plate of dimensions a and b in x_1 and x_2 directions and total thickness $2h$, with piezoelectric film actuators attached to its bottom $x_3 = -h$, and top $x_3 = h$ (Fig. 1) surfaces. The positions of the bottom and top surfaces as well as of the $N-1$ interfaces between the laminates are denoted by $h^{(0)} = -h$, $h^{(1)}$, $h^{(2)}, \ldots, h^{(N-1)}, h^{(N)} = h$; the ith elastic layer is determined by $h^{(i-1)} < x_3 < h^{(i)}$, where $i = 1, 2, \ldots, N$. We use a superscript i in parentheses to indicate quantities of the ith layer. Equations expressing the balance of linear momentum are

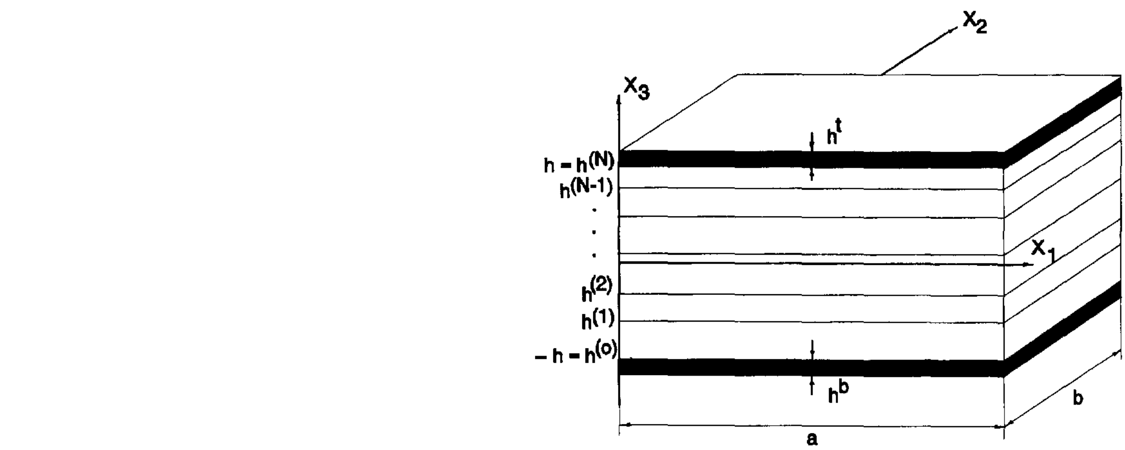

Fig. 1. A N-layer rectangular elastic plate with piezoelectric actuators at the bottom and top.

$$\tau^{(i)}_{\alpha\beta,\beta} = \rho^{(i)}\ddot{u}^{(i)}_{\alpha}, \quad \alpha,\beta = 1,2,3 \tag{1}$$

where $\tau^{(i)}_{\alpha\beta}$ is the stress tensor, $u^{(i)}_{\alpha}$ the displacement vector and $\rho^{(i)}$ the mass density of the material of the ith layer. Here we have used a fixed set of rectangular Cartesian axes, a comma followed by the index β implies partial differentiation with respect to x_β, a superimposed dot signifies partial differentiation with respect to time t, and a repeated index implies summation over the range of the index. Each layer is assumed to be made of an orthotropic material with constitutive equations

$$\begin{aligned}
\tau^{(i)}_{11} &= c^{(i)}_{11}u^{(i)}_{1,1} + c^{(i)}_{12}u^{(i)}_{2,2} + c^{(i)}_{13}u^{(i)}_{3,3} \\
\tau^{(i)}_{22} &= c^{(i)}_{12}u^{(i)}_{1,1} + c^{(i)}_{22}u^{(i)}_{2,2} + c^{(i)}_{23}u^{(i)}_{3,3} \\
\tau^{(i)}_{33} &= c^{(i)}_{13}u^{(i)}_{1,1} + c^{(i)}_{23}u^{(i)}_{2,2} + c^{(i)}_{33}u^{(i)}_{3,3} \\
\tau^{(i)}_{23} &= c^{(i)}_{44}(u^{(i)}_{2,3} + u^{(i)}_{3,2}) \\
\tau^{(i)}_{31} &= c^{(i)}_{55}(u^{(i)}_{3,1} + u^{(i)}_{1,3}) \\
\tau^{(i)}_{12} &= c^{(i)}_{66}(u^{(i)}_{1,2} + u^{(i)}_{2,1})
\end{aligned} \tag{2}$$

which includes isotropic materials as special cases. Here c_{11}, $c_{12}\ldots$, etc. are the elasticities of the orthotropic material. The boundary conditions at the edges of the simply supported plate are taken to be

$$\begin{aligned}
\tau^{(i)}_{11} &= 0, \quad u^{(i)}_3 = 0, \quad u^{(i)}_2 = 0 \quad \text{at} \quad x_1 = 0, a \\
\tau^{(i)}_{22} &= 0, \quad u^{(i)}_3 = 0, \quad u^{(i)}_1 = 0 \quad \text{at} \quad x_2 = 0, b.
\end{aligned} \tag{3}$$

We note that the above boundary conditions are for the case when the plate is viewed as a three-dimensional body. They simulate a simply supported plate characterized by the vanishing of the deflection and bending moment at the edges. Boundary conditions such as eqn (3) have been used by Srinivas *et al.* (1970) and Wittrick (1987) to simulate simply supported plates. At the interface $x_3 = h^{(i)}$ between the ith and the $(i+1)$th layers, we have for $i = 1, 2, \ldots, N-1$

$$\begin{aligned}
\tau^{(i)}_{31} &= \tau^{(i+1)}_{31}, \quad \tau^{(i)}_{32} = \tau^{(i+1)}_{32}, \quad \tau^{(i)}_{33} = \tau^{(i+1)}_{33} \quad \text{at} \quad x_3 = h^{(i)} \\
u^{(i)}_1 &= u^{(i+1)}_1, \quad u^{(i)}_2 = u^{(i+1)}_2, \quad u^{(i)}_3 = u^{(i+1)}_3 \quad \text{at} \quad x_3 = h^{(i)}.
\end{aligned} \tag{4}$$

The substitution of eqn (2) into eqns (1), (3) and (4) yields equations of equilibrium and boundary as well as interface conditions in terms of displacements:

$$\begin{aligned}
&c_{11}^{(i)}u_{1,11}^{(i)}+c_{66}^{(i)}u_{1,22}^{(i)}+c_{55}^{(i)}u_{1,33}^{(i)}+(c_{12}^{(i)}+c_{66}^{(i)})u_{2,12}^{(i)}+(c_{13}^{(i)}+c_{55}^{(i)})u_{3,13}^{(i)} = \rho^{(i)}\ddot{u}_1^{(i)}\\
&(c_{12}^{(i)}+c_{66}^{(i)})u_{1,12}^{(i)}+c_{66}^{(i)}u_{2,11}^{(i)}+c_{22}^{(i)}u_{2,22}^{(i)}+c_{44}^{(i)}u_{2,33}^{(i)}+(c_{23}^{(i)}+c_{44}^{(i)})u_{3,23}^{(i)} = \rho^{(i)}\ddot{u}_2^{(i)}\\
&(c_{13}^{(i)}+c_{55}^{(i)})u_{1,13}^{(i)}+(c_{23}^{(i)}+c_{44}^{(i)})u_{2,23}^{(i)}+c_{55}^{(i)}u_{3,11}^{(i)}+c_{44}^{(i)}u_{3,22}^{(i)}+c_{33}^{(i)}u_{3,33}^{(i)} = \rho^{(i)}\ddot{u}_3^{(i)}\\
&c_{11}^{(i)}u_{1,1}^{(i)}+c_{12}^{(i)}u_{2,2}^{(i)}+c_{13}^{(i)}u_{3,3}^{(i)} = 0, \quad u_3^{(i)} = 0, \quad u_2^{(i)} = 0 \quad \text{at} \quad x_1 = 0, a\\
&c_{12}^{(i)}u_{1,1}^{(i)}+c_{22}^{(i)}u_{2,2}^{(i)}+c_{23}^{(i)}u_{3,3}^{(i)} = 0, \quad u_3^{(i)} = 0, \quad u_1^{(i)} = 0 \quad \text{at} \quad x_2 = 0, b
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
c_{55}^{(i)}(u_{3,1}^{(i)}+u_{1,3}^{(i)}) &= c_{55}^{(i+1)}(u_{3,1}^{(i+1)}+u_{1,3}^{(i+1)}) \quad \text{at} \quad x_3 = h^{(i)}\\
c_{44}^{(i)}(u_{2,3}^{(i)}+u_{3,2}^{(i)}) &= c_{44}^{(i+1)}(u_{2,3}^{(i+1)}+u_{3,2}^{(i+1)}) \quad \text{at} \quad x_3 = h^{(i)}\\
c_{13}^{(i)}u_{1,1}^{(i)}+c_{23}^{(i)}u_{2,2}^{(i)}+c_{33}^{(i)}u_{3,3}^{(i)} &= c_{13}^{(i+1)}u_{1,1}^{(i+1)}+c_{23}^{(i+1)}u_{2,2}^{(i+1)}+c_{33}^{(i+1)}u_{3,3}^{(i+1)} \quad \text{at} \quad x_3 = h^{(i)}\\
\mathrm{u}_1^{(i)} = \mathrm{u}_1^{(i+1)}, \quad \mathrm{u}_2^{(i)} &= \mathrm{u}_2^{(i+1)}, \quad \mathrm{u}_3^{(i)} = \mathrm{u}_3^{(i+1)} \quad \text{at} \quad x_3 = h^{(i)}.
\end{aligned} \tag{6}$$

Usually the piezo ceramic actuators are quite thin as compared to the laminate and therefore can be modeled as thin films (Tiersten, 1993). We use a superscript b to denote quantities for the bottom actuator. For the bottom actuator of thickness h^b, the balance of linear momentum yields

$$h^b\tau_{11,1}^b+h^b\tau_{21,2}^b+\tau_{31}^{(1)}|_{x_3=-h} = \rho^b h^b\ddot{u}_1^b, \quad h^b\tau_{12,1}^b+h^b\tau_{22,2}^b+\tau_{32}^{(1)}|_{x_3=-h} = \rho^b h^b\ddot{u}_2^b. \tag{7}$$

Polarized ceramics, poled in x_3 direction, can be modeled as transversely isotropic materials with the x_3-axis as the preferred direction; their constitutive equations which satisfy the charge equation of electrostatics to the lowest order are (Tiersten, 1993)

$$\begin{aligned}
\tau_{11}^b &= \bar{c}_{11}^b u_{1,1}^b+\bar{c}_{12}^b u_{2,2}^b-\bar{e}_{31}^b E_3^b = \bar{c}_{11}^b u_{1,1}^b+\bar{c}_{12}^b u_{2,2}^b-\bar{e}_{31}^b V^b(t)/h^b\\
\tau_{22}^b &= \bar{c}_{12}^b u_{1,1}^b+\bar{c}_{11}^b u_{2,2}^b-\bar{e}_{31}^b E_3^b = \bar{c}_{12}^b u_{1,1}^b+\bar{c}_{11}^b u_{2,2}^b-\bar{e}_{31}^b V^b(t)/h^b\\
\tau_{12}^b &= c_{66}^b(u_{1,2}^b+u_{2,1}^b)
\end{aligned} \tag{8}$$

where

$$\begin{aligned}
\bar{c}_{11}^b &= c_{11}^b-c_{13}^b c_{13}^b/c_{33}^b = s_{11}^b/\Delta, \quad \bar{c}_{12}^b = c_{12}^b-c_{13}^b c_{13}^b/c_{33}^b = s_{12}^b/\Delta\\
\bar{e}_{31}^b &= e_{31}^b-e_{33}^b c_{13}^b/c_{33}^b = d_{31}^b/(s_{11}^b+s_{12}^b), \quad \Delta = s_{11}^b s_{11}^b-s_{12}^b s_{12}^b
\end{aligned} \tag{9}$$

and we have set $E_3^b = V^b(t)/h^b$ where $V^b(t)$ is the applied voltage which is a function of time t only. The edge conditions for the bottom actuator are

$$\begin{aligned}
\tau_{11}^b = 0, \quad u_2^b = 0 \quad \text{at} \quad x_1 = 0, a\\
\tau_{22}^b = 0, \quad u_1^b = 0 \quad \text{at} \quad x_2 = 0, b.
\end{aligned} \tag{10}$$

In terms of displacements, eqns (7) and (10) become

$$\begin{aligned}
&h^b\bar{c}_{11}^b u_{1,11}^b+h^b c_{66}^b u_{1,22}^b+h^b(\bar{c}_{12}^b+c_{66}^b)u_{2,12}^b+c_{55}^{(1)}(u_{3,1}^{(1)}+u_{1,3}^{(1)})|_{x_3=-h} = \rho^b h^b\ddot{u}_1^b\\
&h^b(\bar{c}_{12}^b+c_{66}^b)u_{1,12}^b+h^b c_{66}^b u_{2,11}^b+h^b\bar{c}_{11}^b u_{2,22}^b+c_{44}^{(1)}(u_{3,2}^{(1)}+u_{2,3}^{(1)})|_{x_3=-h} = \rho^b h^b\ddot{u}_2^b
\end{aligned}$$

$$h^b\bar{c}^b_{11}u^b_{1,1} = \bar{e}^b_{31}V^b(t), \quad u^b_2 = 0 \quad \text{at} \quad x_1 = 0, a$$
$$h^b\bar{c}^b_{11}u^b_{2,2} = \bar{e}^b_{31}V^b(t), \quad u^b_1 = 0 \quad \text{at} \quad x_2 = 0, b. \tag{11}$$

Similarly, for the top actuator with thickness h^t and applied voltage $V^t(t)$ we have

$$h^t\bar{c}^t_{11}u^t_{1,11} + h^tc^t_{66}u^t_{1,22} + h^t(\bar{c}^t_{12} + c^t_{66})u^t_{2,12} - c^{(N)}_{55}(u^{(N)}_{3,1} + u^{(N)}_{1,3})|_{x_3=h} = \rho^th^t\ddot{u}^t_1$$
$$h^t(\bar{c}^t_{12} + c^t_{66})u^t_{1,12} + h^tc^t_{66}u^t_{2,11} + h^t\bar{c}^t_{11}u^t_{2,22} - c^{(N)}_{44}(u^{(N)}_{3,2} + u^{(N)}_{2,3})|_{x_3=h} = \rho^th^t\ddot{u}^t_2$$
$$h^t\bar{c}^t_{11}u^t_{1,1} = \bar{e}^t_{31}V^t(t), \quad u^t_2 = 0 \quad \text{at} \quad x_1 = 0, a$$
$$h^t\bar{c}^t_{11}u^t_{2,2} = \bar{e}^t_{31}V^t(t), \quad u^t_1 = 0 \quad \text{at} \quad x_2 = 0, b, \tag{12}$$

where a superscript t denotes quantities for the top actuator, and we have set $E^t_3 = V^t(t)/h^t$.

The additional boundary conditions at the bottom surface $x_3 = -h$ and top surface $x_3 = h$ of the plate are given by

$$\tau^{(1)}_{33} = \rho^bh^b\ddot{u}^{(1)}_3, \quad u^{(1)}_1 = u^b_1, \quad u^{(1)}_2 = u^b_2 \quad \text{at} \quad x_3 = -h \tag{13}$$
$$-\tau^{(N)}_{33} = \rho^th^t\ddot{u}^{(N)}_3, \quad u^{(N)}_1 = u^t_1, \quad u^{(N)}_2 = u^t_2 \quad \text{at} \quad x_3 = h$$

or, in terms of displacements

$$c^{(1)}_{13}u^{(1)}_{1,1} + c^{(1)}_{23}u^{(1)}_{2,2} + c^{(1)}_{33}u^{(1)}_{3,3} = \rho^bh^b\ddot{u}^{(1)}_3, \quad u^{(1)}_1 = u^b_1, \quad u^{(1)}_2 = u^b_2 \quad \text{at} \quad x_3 = -h$$
$$-(c^{(N)}_{13}u^{(N)}_{1,1} + c^{(N)}_{23}u^{(N)}_{2,2} + c^{(N)}_{33}u^{(N)}_{3,3}) = \rho^th^t\ddot{u}^{(N)}_3, \quad u^{(N)}_1 = u^t_1, \quad u^{(N)}_2 = u^t_2 \quad \text{at} \quad x_3 = h. \tag{14}$$

The problem we need to solve consists of eqns (5), (6), (11), (12) and (14), with unknowns $u^{(i)}_1(x_1, x_2, x_3, t)$, $u^{(i)}_2(x_1, x_2, x_3, t)$, $u^{(i)}_3(x_1, x_2, x_3, t)$, $u^b_1(x_1, x_2, t)$, $u^b_2(x_1, x_2, t)$, $u^t_1(x_1, x_2, t)$ and $u^t_2(x_1, x_2, t)$.

2.2. *Time harmonic vibrations*

We consider vibrations of the plate under time harmonic driving voltage

$$V^b(t) = \bar{V}^b e^{i\omega t}, \quad V^t(t) = \bar{V}^t e^{i\omega t} \tag{15}$$

in which $\bar{V}^b$ and $\bar{V}^t$ are constants, and restrict ourselves to steady state vibrations for which all field quantities have the same time dependence

$$u^{(i)}_\alpha(x_1, x_2, x_3, t) = \tilde{u}^{(i)}_\alpha(x_1, x_2, x_3)e^{i\omega t}, \quad \alpha = 1, 2, 3$$
$$u^b_m(x_1, x_2, t) = \tilde{u}^b_m(x_1, x_2)e^{i\omega t}, \quad m = 1, 2$$
$$u^t_m(x_1, x_2, t) = \tilde{u}^t_m(x_1, x_2)e^{i\omega t}, \quad m = 1, 2. \tag{16}$$

Henceforth, we drop the superimposed tildes. Substitution of eqn (16) into (5), (6), (11), (12) and (14) yields

$$c^{(i)}_{11}u^{(i)}_{1,11} + c^{(i)}_{66}u^{(i)}_{1,22} + c^{(i)}_{55}u^{(i)}_{1,33} + (c^{(i)}_{12} + c^{(i)}_{66})u^{(i)}_{2,12} + (c^{(i)}_{13} + c^{(i)}_{55})u^{(i)}_{3,13} = -\rho^{(i)}\omega^2u^{(i)}_1$$
$$(c^{(i)}_{12} + c^{(i)}_{66})u^{(i)}_{1,12} + c^{(i)}_{66}u^{(i)}_{2,11} + c^{(i)}_{22}u^{(i)}_{2,22} + c^{(i)}_{44}u^{(i)}_{2,33} + (c^{(i)}_{23} + c^{(i)}_{44})u^{(i)}_{3,23} = -\rho^{(i)}\omega^2u^{(i)}_2$$
$$(c^{(i)}_{13} + c^{(i)}_{55})u^{(i)}_{1,13} + (c^{(i)}_{23} + c^{(i)}_{44})u^{(i)}_{2,23} + c^{(i)}_{55}u^{(i)}_{3,11} + c^{(i)}_{44}u^{(i)}_{3,22} + c^{(i)}_{33}u^{(i)}_{3,33} = -\rho^{(i)}\omega^2u^{(i)}_3$$

$$c_{11}^{(i)}u_{1,1}^{(i)}+c_{12}^{(i)}u_{2,2}^{(i)}+c_{13}^{(i)}u_{3,3}^{(i)}=0,\quad u_3^{(i)}=0,\quad u_2^{(i)}=0\quad \text{at}\quad x_1=0,a$$
$$c_{12}^{(i)}u_{1,1}^{(i)}+c_{22}^{(i)}u_{2,2}^{(i)}+c_{23}^{(i)}u_{3,3}^{(i)}=0,\quad u_3^{(i)}=0,\quad u_1^{(i)}=0\quad \text{at}\quad x_2=0,b \tag{17}$$

$$c_{55}^{(i)}(u_{3,1}^{(i)}+u_{1,3}^{(i)})=c_{55}^{(i+1)}(u_{3,1}^{(i+1)}+u_{1,3}^{(i+1)})\quad \text{at}\quad x_3=h^{(i)}$$
$$c_{44}^{(i)}(u_{2,3}^{(i)}+u_{3,2}^{(i)})=c_{44}^{(i+1)}(u_{2,3}^{(i+1)}+u_{3,2}^{(i+1)})\quad \text{at}\quad x_3=h^{(i)}$$
$$c_{13}^{(i)}u_{1,1}^{(i)}+c_{23}^{(i)}u_{2,2}^{(i)}+c_{33}^{(i)}u_{3,3}^{(i)}=c_{13}^{(i+1)}u_{1,1}^{(i+1)}+c_{23}^{(i+1)}u_{2,2}^{(i+1)}+c_{33}^{(i+1)}u_{3,3}^{(i+1)}\quad \text{at}\quad x_3=h^{(i)}$$
$$u_1^{(i)}=u_1^{(i+1)},\quad u_2^{(i)}=u_2^{(i+1)},\quad u_3^{(i)}=u_3^{(i+1)}\quad \text{at}\quad \mathrm{x}_3=\mathrm{h}^{(i)} \tag{18}$$

$$h^b\bar{c}_{11}^b u_{1,11}^b+h^b c_{66}^b u_{1,22}^b+h^b(\bar{c}_{12}^b+c_{66}^b)u_{2,12}^b+c_{55}^{(1)}(u_{3,1}^{(1)}+u_{1,3}^{(1)})|_{x_3=-h}=-\rho^b h^b\omega^2 u_1^b$$
$$h^b(\bar{c}_{12}^b+c_{66}^b)u_{1,12}^b+h^b c_{66}^b u_{2,11}^b+h^b\bar{c}_{11}^b u_{2,22}^b+c_{44}^{(1)}(u_{3,2}^{(1)}+u_{2,3}^{(1)})|_{x_3=-h}=-\rho^b h^b\omega^2 u_2^b$$
$$h^b\bar{c}_{11}^b u_{1,1}^b=\bar{e}_{31}^b\bar{V}^b,\quad u_2^b=0\quad \text{at}\quad x_1=0,a$$
$$h^b\bar{c}_{11}^b u_{2,2}^b=\bar{e}_{31}^b\bar{V}^b,\quad u_1^b=0\quad \text{at}\quad x_2=0,b \tag{19}$$

$$h^t\bar{c}_{11}^t u_{1,11}^t+h^t c_{66}^t u_{1,22}^t+h^t(\bar{c}_{12}^t+c_{66}^t)u_{2,12}^t-c_{55}^{(N)}(u_{3,1}^{(N)}+u_{1,3}^{(N)})|_{x_3=h}=-\rho^t h^t\omega^2 u_1^t$$
$$h^t(\bar{c}_{12}^t+c_{66}^t)u_{1,12}^t+h^t c_{66}^t u_{2,11}^t+h^t\bar{c}_{11}^t u_{2,22}^t-c_{44}^{(N)}(u_{3,2}^{(N)}+u_{2,3}^{(N)})|_{x_3=h}=-\rho^t h^t\omega^2 u_2^t$$
$$h^t\bar{c}_{11}^t u_{1,1}^t=\bar{e}_{31}^t\bar{V}^t,\quad u_2^t=0\quad \text{at}\quad x_1=0,a$$
$$h^t\bar{c}_{11}^t u_{2,2}^t=\bar{e}_{31}^t\bar{V}^t,\quad u_1^t=0\quad \text{at}\quad x_2=0,b \tag{20}$$

$$c_{13}^{(1)}u_{1,1}^{(1)}+c_{23}^{(1)}u_{2,2}^{(1)}+c_{33}^{(1)}u_{3,3}^{(1)}=-\rho^b h^b\omega^2 u_3^{(1)},\quad u_1^{(1)}=u_1^b,\quad u_2^{(1)}=u_2^b\quad \text{at}\quad x_3=-h$$
$$-(c_{13}^{(N)}u_{1,1}^{(N)}+c_{23}^{(N)}u_{2,2}^{(N)}+c_{33}^{(N)}u_{3,3}^{(N)})=-\rho^t h^t\omega^2 u_3^{(N)},\quad u_1^{(N)}=u_1^t,\quad u_2^{(N)}=u_2^t\quad \text{at}\quad x_3=h. \tag{21}$$

3. SOLUTIONS

3.1. *Solutions for the laminates*

We assume that displacements of the ith layer can be represented as Fourier series. That is

$$u_1^{(i)}=\sum_{m,n=1}^{\infty}a_{mn}^{(i)}(x_3)\cos\alpha_m x_1\sin\beta_n x_2$$
$$u_2^{(i)}=\sum_{m,n=1}^{\infty}b_{mn}^{(i)}(x_3)\sin\alpha_m x_1\cos\beta_n x_2$$
$$u_3^{(i)}=\sum_{m,n=1}^{\infty}c_{mn}^{(i)}(x_3)\sin\alpha_m x_1\sin\beta_n x_2$$
$$\alpha_m=m\pi/a,\quad \beta_n=n\pi/b \tag{22}$$

which ensures that all homogeneous boundary conditions in eqn (17) at the edges $x_1=0$, a and $x_2=0,b$ are satisfied. The substitution of eqn (22) into eqn $(17)_{1-3}$ yields the following ordinary differential equations

$$-c_{11}^{(i)}\alpha_m^2 a_{mn}^{(i)}-c_{66}^{(i)}\beta_n^2 a_{mn}^{(i)}+c_{55}^{(i)}a_{mn,33}^{(i)}-(c_{12}^{(i)}+c_{66}^{(i)})\alpha_m\beta_n b_{mn}^{(i)}+(c_{13}^{(i)}+c_{55}^{(i)})\alpha_m c_{mn,3}^{(i)}=-\rho^{(i)}\omega^2 a_{mn}^{(i)}$$
$$-(c_{12}^{(i)}+c_{66}^{(i)})\alpha_m\beta_n a_{mn}^{(i)}-c_{66}^{(i)}\alpha_m^2 b_{mn}^{(i)}-c_{22}^{(i)}\beta_n^2 b_{mn}^{(i)}+c_{44}^{(i)}b_{mn,33}^{(i)}+(c_{23}^{(i)}+c_{44}^{(i)})\beta_n c_{mn,3}^{(i)}=-\rho^{(i)}\omega^2 b_{mn}^{(i)}$$
$$-(c_{13}^{(i)}+c_{55}^{(i)})\alpha_m a_{mn,3}^{(i)}-(c_{23}^{(i)}+c_{44}^{(i)})\beta_n b_{mn,3}^{(i)}-c_{55}^{(i)}\alpha_m^2 c_{mn}^{(i)}-c_{44}^{(i)}\beta_n^2 c_{mn}^{(i)}+c_{33}^{(i)}c_{mn,33}^{(i)}=-\rho^{(i)}\omega^2 c_{mn}^{(i)}. \tag{23}$$

In order to find a solution of (23), we let

$$
\begin{aligned}
a_{mn}^{(i)}(x_3) &= A_{mn}^{(i)} e^{\eta_{mn}^{(i)} x_3} \\
b_{mn}^{(i)}(x_3) &= B_{mn}^{(i)} e^{\eta_{mn}^{(i)} x_3} \\
c_{mn}^{(i)}(x_3) &= C_{mn}^{(i)} e^{\eta_{mn}^{(i)} x_3}
\end{aligned} \tag{24}
$$

where $A_{mn}^{(i)}$, $B_{mn}^{(i)}$, $C_{mn}^{(i)}$ and $\eta_{mn}^{(i)}$ are undetermined constants. Substitution of eqn (24) into eqn (23) yields the following homogeneous linear equations for the determination of $A_{mn}^{(i)}$, $B_{mn}^{(i)}$ and $C_{mn}^{(i)}$

$$
\begin{aligned}
&[\rho^{(i)}\omega^2 - c_{11}^{(i)}\alpha_m^2 - c_{66}^{(i)}\beta_n^2 + c_{55}^{(i)}(\eta_{mn}^{(i)})^2]A_{mn}^{(i)} - (c_{12}^{(i)} + c_{66}^{(i)})\alpha_m\beta_n B_{mn}^{(i)} + (c_{13}^{(i)} + c_{55}^{(i)})\alpha_m\eta_{mn}^{(i)} C_{mn}^{(i)} = 0 \\
&-(c_{12}^{(i)} + c_{66}^{(i)})\alpha_m\beta_n A_{mn}^{(i)} + [\rho^{(i)}\omega^2 - c_{66}^{(i)}\alpha_m^2 - c_{22}^{(i)}\beta_n^2 + c_{44}^{(i)}(\eta_{mn}^{(i)})^2]B_{mn}^{(i)} + (c_{23}^{(i)} + c_{44}^{(i)})\beta_n\eta_{mn}^{(i)} C_{mn}^{(i)} = 0 \\
&(c_{13}^{(i)} + c_{55}^{(i)})\alpha_m\eta_{mn}^{(i)} A_{mn}^{(i)} - (c_{23}^{(i)} + c_{44}^{(i)})\beta_n\eta_{mn}^{(i)} B_{mn}^{(i)} - [\rho^{(i)}\omega^2 - c_{55}^{(i)}\alpha_m^2 - c_{44}^{(i)}\beta_n^2 + c_{33}^{(i)}(\eta_{mn}^{(i)})^2]C_{mn}^{(i)} = 0
\end{aligned} \tag{25}
$$

or

$$
\begin{aligned}
&[c_{55}^{(i)}(\eta_{mn}^{(i)})^2 - \lambda_{11}]A_{mn}^{(i)} - \lambda_{12}B_{mn}^{(i)} + \lambda_{13}\eta_{mn}^{(i)}C_{mn}^{(i)} = 0 \\
&-\lambda_{12}A_{mn}^{(i)} + [c_{44}^{(i)}(\eta_{mn}^{(i)})^2 - \lambda_{22}]B_{mn}^{(i)} + \lambda_{23}\eta_{mn}^{(i)}C_{mn}^{(i)} = 0 \\
&-\lambda_{13}\eta_{mn}^{(i)}A_{mn}^{(i)} - \lambda_{23}\eta_{mn}^{(i)}B_{mn}^{(i)} + [c_{33}^{(i)}(\eta_{mn}^{(i)})^2 - \lambda_{33}]C_{mn}^{(i)} = 0
\end{aligned} \tag{26}
$$

where

$$
\begin{aligned}
&\lambda_{11} = c_{11}^{(i)}\alpha_m^2 + c_{66}^{(i)}\beta_n^2 - \rho^{(i)}\omega^2, \quad \lambda_{22} = c_{66}^{(i)}\alpha_m^2 + c_{22}^{(i)}\beta_n^2 - \rho^{(i)}\omega^2, \\
&\lambda_{33} = c_{55}^{(i)}\alpha_m^2 + c_{44}^{(i)}\beta_n^2 - \rho^{(i)}\omega^2, \quad \lambda_{12} = (c_{12}^{(i)} + c_{66}^{(i)})\alpha_m\beta_n, \\
&\lambda_{13} = (c_{13}^{(i)} + c_{55}^{(i)})\alpha_m, \quad \lambda_{23} = (c_{23}^{(i)} + c_{44}^{(i)})\beta_n.
\end{aligned} \tag{27}
$$

Note that all the lambdas should also have the subscripts *mn* and superscript *i* in parentheses—we omit them for simplicity. Equations (26) have nontrivial solutions only if the determinant of the coefficients matrix vanishes. This results in the following cubic equation for $(\eta_{mn}^{(i)})^2$

$$
(\eta_{mn}^{(i)})^6 + a(\eta_{mn}^{(i)})^4 + b(\eta_{mn}^{(i)})^2 + c = 0 \tag{28}
$$

where

$$
\begin{aligned}
a &= \frac{1}{c_{33}^{(i)}c_{44}^{(i)}c_{55}^{(i)}}(c_{44}^{(i)}\lambda_{13}^2 + c_{55}^{(i)}\lambda_{23}^2 - c_{33}^{(i)}c_{44}^{(i)}\lambda_{11} - c_{33}^{(i)}c_{55}^{(i)}\lambda_{22} - c_{44}^{(i)}c_{55}^{(i)}\lambda_{33}) \\
b &= \frac{1}{c_{33}^{(i)}c_{44}^{(i)}c_{55}^{(i)}}(2\lambda_{12}\lambda_{13}\lambda_{23} - \lambda_{23}^2\lambda_{11} - \lambda_{13}^2\lambda_{22} - \lambda_{12}^2c_{33}^{(i)} + c_{33}^{(i)}\lambda_{11}\lambda_{22} + c_{55}^{(i)}\lambda_{22}\lambda_{33} + c_{44}^{(i)}\lambda_{11}\lambda_{33}) \\
c &= \frac{1}{c_{33}^{(i)}c_{44}^{(i)}c_{55}^{(i)}}(\lambda_{12}^2\lambda_{33} - \lambda_{11}\lambda_{22}\lambda_{33}).
\end{aligned} \tag{29}
$$

With

$$y = (\eta_{mn}^{(i)})^2 + \frac{a}{3} \tag{30}$$

$$p = b - \tfrac{1}{3}a^2, \quad q = \tfrac{2}{27}a^3 - \tfrac{1}{3}ab + c \tag{31}$$

the three roots of (28) are given by (Wang, 1987)

$$\begin{aligned}
y_1 &= \left\{-\frac{q}{2} + \left[\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3\right]^{1/2}\right\}^{1/3} + \left\{-\frac{q}{2} - \left[\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3\right]^{1/2}\right\}^{1/3} \\
y_2 &= \omega_1\left\{-\frac{q}{2} + \left[\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3\right]^{1/2}\right\}^{1/3} + \omega_2\left\{-\frac{q}{2} - \left[\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3\right]^{1/2}\right\}^{1/3} \\
y_3 &= \omega_2\left\{-\frac{q}{2} + \left[\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3\right]^{1/2}\right\}^{1/3} + \omega_1\left\{-\frac{q}{2} - \left[\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3\right]^{1/2}\right\}^{1/3}
\end{aligned} \tag{32}$$

where

$$\omega_1 = \tfrac{1}{2}(-1+\sqrt{3}\mathrm{i}), \quad \omega_2 = \tfrac{1}{2}(-1-\sqrt{3}\mathrm{i}). \tag{33}$$

Corresponding to eqn (32) there are six roots for $\eta_{mn}^{(i)}$, which can be real or complex. When a particular root $\eta_{mnp}^{(i)}$ with fixed p is real, from eqn (26) (lines 1 and 2) we obtain that the solution to eqn (23) corresponding to this particular $\eta_{mnp}^{(i)}$ is

$$\begin{aligned}
a_{mn}^{(i)}(x_3) &= D_{mnp}^{(i)} F_{mnp}^{(i)}(x_3) \\
b_{mn}^{(i)}(x_3) &= D_{mnp}^{(i)} G_{mnp}^{(i)}(x_3) \\
c_{mn}^{(i)}(x_3) &= D_{mnp}^{(i)} H_{mnp}^{(i)}(x_3)
\end{aligned} \tag{34}$$

where $D_{mnp}^{(i)}$ is an arbitrary constant, and

$$\begin{aligned}
F_{mnp}^{(i)}(x_3) &= \begin{vmatrix} -\lambda_{13}\eta_{mnp}^{(i)} & -\lambda_{12} \\ -\lambda_{23}\eta_{mnp}^{(i)} & c_{44}^{(i)}(\eta_{mnp}^{(i)})^2 - \lambda_{22} \end{vmatrix} e^{\eta_{mnp}^{(i)}x_3} \\
G_{mnp}^{(i)}(x_3) &= \begin{vmatrix} c_{55}^{(i)}(\eta_{mnp}^{(i)})^2 - \lambda_{11} & -\lambda_{13}\eta_{mnp}^{(i)} \\ -\lambda_{12} & -\lambda_{23}\eta_{mnp}^{(i)} \end{vmatrix} e^{\eta_{mnp}^{(i)}x_3} \\
H_{mnp}^{(i)}(x_3) &= \begin{vmatrix} c_{55}^{(i)}(\eta_{mnp}^{(i)})^2 - \lambda_{11} & -\lambda_{12} \\ -\lambda_{12} & c_{44}^{(i)}(\eta_{mnp}^{(i)})^2 - \lambda_{22} \end{vmatrix} e^{\eta_{mnp}^{(i)}x_3}.
\end{aligned} \tag{35}$$

When a particular $\eta_{mnp}^{(i)}$ is complex, its complex conjugate is also a root. We write one of this pair of roots as $\eta_{mnp}^{(i)} = \xi_{mnp}^{(i)} + \mathrm{i}\zeta_{mnp}^{(i)}$, where $\xi_{mnp}^{(i)}$ and $\zeta_{mnp}^{(i)}$ are real and $\xi_{mnp}^{(i)}$ may be zero but $\zeta_{mnp}^{(i)}$ may not. From eqn (26) (lines 1 and 2) there are two sets of solutions to eqn (23) corresponding to this pair of complex conjugate roots. They all can be written in the form of eqn (34), with

$$\begin{aligned}
F_{mnp}^{(i)}(x_3) &= \cos\zeta_{mnp}^{(i)}x_3 e^{\xi_{mnp}^{(i)}x_3} \\
G_{mnp}^{(i)}(x_3) &= (\gamma_{11}\cos\zeta_{mnp}^{(i)}x_3 - \gamma_{21}\sin\zeta_{mnp}^{(i)}x_3)e^{\xi_{mnp}^{(i)}x_3} \\
H_{mnp}^{(i)}(x_3) &= (\gamma_{31}\cos\zeta_{mnp}^{(i)}x_3 - \gamma_{41}\sin\zeta_{mnp}^{(i)}x_3)e^{\xi_{mnp}^{(i)}x_3}
\end{aligned} \tag{36}$$

and

$$F^{(i)}_{mnp}(x_3) = \sin\zeta^{(i)}_{mnp}x_3 e^{\xi^{(i)}_{mnp}x_3}$$

$$G^{(i)}_{mnp}(x_3) = (-\gamma_{12}\cos\zeta^{(i)}_{mnp}x_3 + \gamma_{22}\sin\zeta^{(i)}_{mnp}x_3)e^{\xi^{(i)}_{mnp}x_3}$$

$$H^{(i)}_{mnp}(x_3) = (-\gamma_{32}\cos\zeta^{(i)}_{mnp}x_3 + \gamma_{42}\sin\zeta^{(i)}_{mnp}x_3)e^{\xi^{(i)}_{mnp}x_3} \tag{37}$$

respectively, where

$$\begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \\ \gamma_{31} & \gamma_{32} \\ \gamma_{41} & \gamma_{42} \end{bmatrix} = \begin{bmatrix} \lambda_{12} & 0 & -\lambda_{13}\xi^{(i)}_{mnp} & \lambda_{13}\zeta^{(i)}_{mnp} \\ 0 & \lambda_{12} & -\lambda_{13}\zeta^{(i)}_{mnp} & -\lambda_{13}\xi^{(i)}_{mnp} \\ c_{44}[(\xi^{(i)}_{mnp})^2-(\zeta^{(i)}_{mnp})^2]-\lambda_{22} & -2c_{44}\xi^{(i)}_{mnp}\zeta^{(i)}_{mnp} & \lambda_{23}\xi^{(i)}_{mnp} & -\lambda_{23}\zeta^{(i)}_{mnp} \\ 2c_{44}\xi^{(i)}_{mnp}\zeta^{(i)}_{mnp} & c_{44}[(\xi^{(i)}_{mnp})^2-(\zeta^{(i)}_{mnp})^2]-\lambda_{22} & \lambda_{23}\zeta^{(i)}_{mnp} & \lambda_{23}\xi^{(i)}_{mnp} \end{bmatrix}^{-1}$$

$$\times \begin{bmatrix} c_{55}[(\xi^{(i)}_{mnp})^2-(\zeta^{(i)}_{mnp})^2]-\lambda_{11} & -2c_{55}\xi^{(i)}_{mnp}\zeta^{(i)}_{mnp} \\ 2c_{55}\xi^{(i)}_{mnp}\zeta^{(i)}_{mnp} & c_{55}[(\xi^{(i)}_{mnp})^2-(\zeta^{(i)}_{mnp})^2]-\lambda_{11} \\ \lambda_{12} & 0 \\ 0 & \lambda_{12} \end{bmatrix}. \tag{38}$$

We note that there can be a few values of ω which make $q^2/4+p^3/27 = 0$. In that case, eqn (28) has repeated roots for $(\eta^{(i)}_{mn})^2$ and the solution to eqn (23) needs special discussion. This is likely to be the case when each of the laminate is made of an isotropic material. We will not consider these special values of ω. The general solution to eqn (23) can therefore be written as

$$a^{(i)}_{mn}(x_3) = \sum_{p=1}^{6} D^{(i)}_{mnp}F^{(i)}_{mnp}(x_3)$$

$$b^{(i)}_{mn}(x_3) = \sum_{p=1}^{6} D^{(i)}_{mnp}G^{(i)}_{mnp}(x_3)$$

$$c^{(i)}_{mn}(x_3) = \sum_{p=1}^{6} D^{(i)}_{mnp}H^{(i)}_{mnp}(x_3) \tag{39}$$

where $D^{(i)}_{mnp}$ are undetermined constants. The displacements and stresses can then be written as

$$u^{(i)}_1 = \sum_{m,n=1}^{\infty}\left[\sum_{p=1}^{6} D^{(i)}_{mnp}F^{(i)}_{mnp}(x_3)\right]\cos\alpha_m x_1 \sin\beta_n x_2$$

$$u^{(i)}_2 = \sum_{m,n=1}^{\infty}\left[\sum_{p=1}^{6} D^{(i)}_{mnp}G^{(i)}_{mnp}(x_3)\right]\sin\alpha_m x_1 \cos\beta_n x_2$$

$$u^{(i)}_3 = \sum_{m,n=1}^{\infty}\left[\sum_{p=1}^{6} D^{(i)}_{mnp}H^{(i)}_{mnp}(x_3)\right]\sin\alpha_m x_1 \sin\beta_n x_2$$

$$\tau^{(i)}_{11} = \sum_{m,n=1}^{\infty}\left[\sum_{p=1}^{6} D^{(i)}_{mnp}K^{(i)}_{mnp}(x_3)\right]\sin\alpha_m x_1 \sin\beta_n x_2$$

$$\tau_{22}^{(i)} = \sum_{m,n=1}^{\infty}\left[\sum_{p=1}^{6} D_{mnp}^{(i)} L_{mnp}^{(i)}(x_3)\right]\sin\alpha_m x_1 \sin\beta_n x_2$$

$$\tau_{33}^{(i)} = \sum_{m,n=1}^{\infty}\left[\sum_{p=1}^{6} D_{mnp}^{(i)} P_{mnp}^{(i)}(x_3)\right]\sin\alpha_m x_1 \sin\beta_n x_2$$

$$\tau_{23}^{(i)} = \sum_{m,n=1}^{\infty}\left[\sum_{p=1}^{6} D_{mnp}^{(i)} Q_{mnp}^{(i)}(x_3)\right]\sin\alpha_m x_1 \cos\beta_n x_2$$

$$\tau_{31}^{(i)} = \sum_{m,n=1}^{\infty}\left[\sum_{p=1}^{6} D_{mnp}^{(i)} R_{mnp}^{(i)}(x_3)\right]\cos\alpha_m x_1 \sin\beta_n x_2$$

$$\tau_{12}^{(i)} = \sum_{m,n=1}^{\infty}\left[\sum_{p=1}^{6} D_{mnp}^{(i)} T_{mnp}^{(i)}(x_3)\right]\cos\alpha_m x_1 \cos\beta_n x_2 \quad (40)$$

where

$$K_{mnp}^{(i)}(x_3) = -c_{11}^{(i)}\alpha_m F_{mnp}^{(i)}(x_3) - c_{12}^{(i)}\beta_n G_{mnp}^{(i)}(x_3) + c_{13}^{(i)} H_{mnp,3}^{(i)}(x_3)$$

$$L_{mnp}^{(i)}(x_3) = -c_{12}^{(i)}\alpha_m F_{mnp}^{(i)}(x_3) - c_{22}^{(i)}\beta_n G_{mnp}^{(i)}(x_3) + c_{23}^{(i)} H_{mnp,3}^{(i)}(x_3)$$

$$P_{mnp}^{(i)}(x_3) = -c_{13}^{(i)}\alpha_m F_{mnp}^{(i)}(x_3) - c_{23}^{(i)}\beta_n G_{mnp}^{(i)}(x_3) + c_{33}^{(i)} H_{mnp,3}^{(i)}(x_3)$$

$$Q_{mnp}^{(i)}(x_3) = c_{44}^{(i)}[\beta_n H_{mnp}^{(i)}(x_3) + G_{mnp,3}^{(i)}(x_3)]$$

$$R_{mp}^{(i)}(x_3) = c_{55}^{(i)}[\alpha_m H_{mnp}^{(i)}(x_3) + F_{mnp,3}^{(i)}(x_3)]$$

$$T_{mnp}^{(i)}(x_3) = c_{66}^{(i)}[\beta_n F_{mnp}^{(i)}(x_3) + \alpha_m G_{mnp}^{(i)}(x_3)]. \quad (41)$$

Substitution of eqn (40) into the interface continuity conditions [eqn (18)] at $x_3 = h^{(i)}$ for $i = 1, 2, \ldots, N-1$ yields

$$\sum_{p=1}^{6} D_{mnp}^{(i)} F_{mnp}^{(i)}(h^{(i)}) = \sum_{p=1}^{6} D_{mnp}^{(i+1)} F_{mnp}^{(i+1)}(h^{(i)})$$

$$\sum_{p=1}^{6} D_{mnp}^{(i)} G_{mnp}^{(i)}(h^{(i)}) = \sum_{p=1}^{6} D_{mnp}^{(i+1)} G_{mnp}^{(i+1)}(h^{(i)})$$

$$\sum_{p=1}^{6} D_{mnp}^{(i)} H_{mnp}^{(i)}(h^{(i)}) = \sum_{p=1}^{6} D_{mnp}^{(i+1)} H_{mnp}^{(i+1)}(h^{(i)})$$

$$\sum_{p=1}^{6} D_{mnp}^{(i)} R_{mnp}^{(i)}(h^{(i)}) = \sum_{p=1}^{6} D_{mnp}^{(i+1)} R_{mnp}^{(i+1)}(h^{(i)})$$

$$\sum_{p=1}^{6} D_{mnp}^{(i)} Q_{mnp}^{(i)}(h^{(i)}) = \sum_{p=1}^{6} D_{mnp}^{(i+1)} Q_{mnp}^{(i+1)}(h^{(i)})$$

$$\sum_{p=1}^{6} D_{mnp}^{(i)} P_{mnp}^{(i)}(h^{(i)}) = \sum_{p=1}^{6} D_{mnp}^{(i+1)} P_{mnp}^{(i+1)}(h^{(i)}) \quad (42)$$

which can be written in the following matrix form

$$\begin{bmatrix} D_{mn1}^{(i)} \\ D_{mn2}^{(i)} \\ D_{mn3}^{(i)} \\ D_{mn4}^{(i)} \\ D_{mn5}^{(i)} \\ D_{mn6}^{(i)} \end{bmatrix} = [T]^{(i)} \begin{bmatrix} D_{mn1}^{(i+1)} \\ D_{mn2}^{(i+1)} \\ D_{mn3}^{(i+1)} \\ D_{mn4}^{(i+1)} \\ D_{mn5}^{(i+1)} \\ D_{mn6}^{(i+1)} \end{bmatrix} \quad (43)$$

where $[T]^{(i)}$ is a transfer matrix given by

$$[T]^{(i)} = \begin{bmatrix} F^{(i)}_{mn1}(h^{(i)}) & F^{(i)}_{mn2}(h^{(i)}) & F^{(i)}_{mn3}(h^{(i)}) & F^{(i)}_{mn4}(h^{(i)}) & F^{(i)}_{mn5}(h^{(i)}) & F^{(i)}_{mn6}(h^{(i)}) \\ G^{(i)}_{mn1}(h^{(i)}) & G^{(i)}_{mn2}(h^{(i)}) & G^{(i)}_{mn3}(h^{(i)}) & G^{(i)}_{mn4}(h^{(i)}) & G^{(i)}_{mn5}(h^{(i)}) & G^{(i)}_{mn6}(h^{(i)}) \\ H^{(i)}_{mn1}(h^{(i)}) & H^{(i)}_{mn2}(h^{(i)}) & H^{(i)}_{mn3}(h^{(i)}) & H^{(i)}_{mn4}(h^{(i)}) & H^{(i)}_{mn5}(h^{(i)}) & H^{(i)}_{mn6}(h^{(i)}) \\ R^{(i)}_{mn1}(h^{(i)}) & R^{(i)}_{mn2}(h^{(i)}) & R^{(i)}_{mn3}(h^{(i)}) & R^{(i)}_{mn4}(h^{(i)}) & R^{(i)}_{mn5}(h^{(i)}) & R^{(i)}_{mn6}(h^{(i)}) \\ Q^{(i)}_{mn1}(h^{(i)}) & Q^{(i)}_{mn2}(h^{(i)}) & Q^{(i)}_{mn3}(h^{(i)}) & Q^{(i)}_{mn4}(h^{(i)}) & Q^{(i)}_{mn5}(h^{(i)}) & Q^{(i)}_{mn6}(h^{(i)}) \\ P^{(i)}_{mn1}(h^{(i)}) & P^{(i)}_{mn2}(h^{(i)}) & P^{(i)}_{mn3}(h^{(i)}) & P^{(i)}_{mn4}(h^{(i)}) & P^{(i)}_{mn5}(h^{(i)}) & P^{(i)}_{mn6}(h^{(i)}) \end{bmatrix}^{-1}$$

$$\times \begin{bmatrix} F^{(i+1)}_{mn1}(h^{(i)}) & F^{(i+1)}_{mn2}(h^{(i)}) & F^{(i+1)}_{mn3}(h^{(i)}) & F^{(i+1)}_{mn4}(h^{(i)}) & F^{(i+1)}_{mn5}(h^{(i)}) & F^{(i+1)}_{mn6}(h^{(i)}) \\ G^{(i+1)}_{mn1}(h^{(i)}) & G^{(i+1)}_{mn2}(h^{(i)}) & G^{(i+1)}_{mn3}(h^{(i)}) & G^{(i+1)}_{mn4}(h^{(i)}) & G^{(i+1)}_{mn5}(h^{(i)}) & G^{(i+1)}_{mn6}(h^{(i)}) \\ H^{(i+1)}_{mn1}(h^{(i)}) & H^{(i+1)}_{mn2}(h^{(i)}) & H^{(i+1)}_{mn3}(h^{(i)}) & H^{(i+1)}_{mn4}(h^{(i)}) & H^{(i+1)}_{mn5}(h^{(i)}) & H^{(i+1)}_{mn6}(h^{(i)}) \\ R^{(i+1)}_{mn1}(h^{(i)}) & R^{(i+1)}_{mn2}(h^{(i)}) & R^{(i+1)}_{mn3}(h^{(i)}) & R^{(i+1)}_{mn4}(h^{(i)}) & R^{(i+1)}_{mn5}(h^{(i)}) & R^{(i+1)}_{mn6}(h^{(i)}) \\ Q^{(i+1)}_{mn1}(h^{(i)}) & Q^{(i+1)}_{mn2}(h^{(i)}) & Q^{(i+1)}_{mn3}(h^{(i)}) & Q^{(i+1)}_{mn4}(h^{(i)}) & Q^{(i+1)}_{mn5}(h^{(i)}) & Q^{(i+1)}_{mn6}(h^{(i)}) \\ P^{(i+1)}_{mn1}(h^{(i)}) & P^{(i+1)}_{mn2}(h^{(i)}) & P^{(i+1)}_{mn3}(h^{(i)}) & P^{(i+1)}_{mn4}(h^{(i)}) & P^{(i+1)}_{mn5}(h^{(i)}) & P^{(i+1)}_{mn6}(h^{(i)}) \end{bmatrix}. \quad (44)$$

With repeated use of eqn (43), we obtain

$$\begin{bmatrix} D^{(1)}_{mn1} \\ D^{(1)}_{mn2} \\ D^{(1)}_{mn3} \\ D^{(1)}_{mn4} \\ D^{(1)}_{mn5} \\ D^{(1)}_{mn6} \end{bmatrix} = [T]^{(1)}[T]^{(2)}\ldots[T]^{(N-1)} \begin{bmatrix} D^{(N)}_{mn1} \\ D^{(N)}_{mn2} \\ D^{(N)}_{mn3} \\ D^{(N)}_{mn4} \\ D^{(N)}_{mn5} \\ D^{(N)}_{mn6} \end{bmatrix} \quad (45)$$

which will be needed later. With eqns (40) and (45), eqns (17) and (18) are satisfied. However, eqns (19)–(21) still remain to be satisfied.

3.2. *Solutions for the actuators*

For the bottom actuator governed by eqn (19), we assume that

$$u^b_1 = \sum_{m,n=1}^{\infty} D^b_{1mn} \cos\alpha_m x_1 \sin\beta_n x_2$$

$$u^b_2 = \sum_{m,n=1}^{\infty} D^b_{2mn} \sin\alpha_m x_1 \cos\beta_n x_2$$

$$D^b_{1mn} = \frac{4}{ab}\int_0^a\int_0^b u^b_1(x_1,x_2)\cos\alpha_m x_1 \sin\beta_n x_2 \, dx_1 \, dx_2$$

$$D^b_{2mn} = \frac{4}{ab}\int_0^a\int_0^b u^b_2(x_1,x_2)\sin\alpha_m x_1 \cos\beta_n x_2 \, dx_1 \, dx_2 \quad (46)$$

where D^b_{1mn} and D^b_{2mn} are undetermined constants. Equations (46) satisfy the two homogeneous displacement boundary conditions in eqn (19). It may seem that, after term by term differentiation, eqn (46) can not accommodate the two nonhomogeneous boundary conditions in eqn (19). This is not the case because the Fourier series [eqn (46)] for this situation with nonhomogeneous boundary conditions generally do not have the uniform convergence needed for term by term differentiation (Zauderer, 1983). To avoid term by term differentiation and to take care of the nonhomogeneous boundary conditions at the same time, we multiply eqn $(19)_1$ by $\cos\alpha_m x_1 \sin\beta_n x_2$ and integrate the resulting equation over $0 < x_1 < a$ and $0 < x_2 < b$. With integration by parts we obtain

$$\int_0^a\int_0^b u^b_{1,11}\cos\alpha_m x_1\sin\beta_n x_2\,dx_1\,dx_2 = -\alpha_m^2\frac{ab}{4}D^b_{1mn} - \frac{\bar{e}^b_{31}\bar{V}^b}{h^b\bar{c}^b_{11}\beta_n}[(-1)^m-1][(-1)^n-1]$$

$$\int_0^a\int_0^b u^b_{1,22}\cos\alpha_m x_1\sin\beta_n x_2\,dx_1\,dx_2 = -\beta_n^2\frac{ab}{4}D^b_{1mn}$$

$$\int_0^a\int_0^b u^b_{2,12}\cos\alpha_m x_1\sin\beta_n x_2\,dx_1\,dx_2 = -\alpha_m\beta_n\frac{ab}{4}D^b_{2mn}$$

$$\int_0^a\int_0^b c^{(1)}_{55}(u^{(1)}_{3,1}+u^{(1)}_{1,3})|_{x_3=-h}\cos\alpha_m x_1\sin\beta_n x_2\,dx_1\,dx_2 = \frac{ab}{4}\left[\sum_{p=1}^{6} R^{(1)}_{mnp}(-h)D^{(1)}_{mnp}\right] \quad (47)$$

where the nonhomogeneous boundary condition in $(19)_3$ has been used. Equation $(19)_1$ then becomes

$$\sum_{p=1}^{6} R^{(1)}_{mnp}(-h)D^{(1)}_{mnp} + h^b(\rho^b\omega^2 - \bar{c}^b_{11}\alpha_m^2 - c^b_{66}\beta_n^2)D^b_{1mn}$$
$$-h^b(\bar{c}^b_{12}+c^b_{66})\alpha_m\beta_n D^b_{2mn} = \frac{4\bar{e}^b_{31}\bar{V}^b}{ab\beta_n}[(-1)^m-1][(-1)^n-1]. \quad (48)$$

In a similar way, we multiply eqn$(19)_2$ by $\sin\alpha_m x_1\cos\beta_n x_2$ and integrate the resulting equation over $0 < x_1 < a$ and $0 < x_2 < b$ to obtain

$$\sum_{p=1}^{6} Q^{(1)}_{mnp}(-h)D^{(1)}_{mnp} - h^b(\bar{c}^b_{12}+c^b_{66})\alpha_m\beta_n D^b_{1mn}$$
$$+h^b(\rho^b\omega^2 - c^b_{66}\alpha_m^2 - \bar{c}^b_{11}\beta_n^2)D^b_{2mn} = \frac{4\bar{e}^b_{31}\bar{V}^b}{ab\alpha_m}[(-1)^m-1][(-1)^n-1] \quad (49)$$

where the nonhomogeneous boundary condition in eqn $(19)_4$ has been used. For the top actuator, we assume

$$u^t_1 = \sum_{m,n=1}^{\infty} D^t_{1mn}\cos\alpha_m x_1\sin\beta_n x_2$$
$$u^t_2 = \sum_{m,n=1}^{\infty} D^t_{2mn}\sin\alpha_m x_1\cos\beta_n x_2$$
$$D^t_{1mn} = \frac{4}{ab}\int_0^a\int_0^b u^t_1(x_1,x_2)\cos\alpha_m x_1\sin\beta_n x_2\,dx_1\,dx_2 \quad (50)$$
$$D^t_{2mn} = \frac{4}{ab}\int_0^a\int_0^b u^t_2(x_1,x_2)\sin\alpha_m x_1\cos\beta_n x_2\,dx_1\,dx_2$$

which satisfy the two homogeneous boundary conditions in eqn $(20)_{3,4}$. By employing the technique similar to that used to obtain eqns (48) and (49), the remaining expressions in eqn (20) are satisfied if

$$\sum_{p=1}^{6} R^{(N)}_{mnp}(h)D^{(N)}_{mnp}+h^t(\bar{c}^t_{11}\alpha_m^2+c^t_{66}\beta_n^2-\rho^t\omega^2)D^t_{1mn}+h^t(\bar{c}^t_{12}+c^t_{66})\alpha_m\beta_n D^t_{2mn}$$

$$=-\frac{4\bar{e}^t_{31}\bar{V}^t}{ab\beta_n}[(-1)^m-1][(-1)^n-1]$$

$$\sum_{p=1}^{6} Q^{(N)}_{mnp}(h)D^{(N)}_{mnp}+h^t(\bar{c}^t_{12}+c^t_{66})\alpha_m\beta_n D^t_{1mn}+h^t(c^t_{66}\alpha_m^2+\bar{c}^t_{11}\beta_n^2-\rho^t\omega^2)D^t_{2mn}$$

$$=-\frac{4\bar{e}^t_{31}\bar{V}^t}{ab\alpha_m}[(-1)^m-1][(-1)^n-1]. \quad (51)$$

With eqn (46) and eqns (48)–(51) all of the expressions in eqns (19) and (20) are satisfied. We are left with eqn (21) only.

3.3. *Continuity conditions between the plate and actuators*

The continuity conditions [eqn (21)] at the interfaces between the plate and the actuators become, upon substitution of eqns (40), (46), and (50)

$$\sum_{p=1}^{6}[P^{(1)}_{mnp}(-h)+\rho^b h^b\omega^2 H^{(1)}_{mnp}(-h)]D^{(1)}_{mnp}=0$$

$$\sum_{p=1}^{6} F^{(1)}_{mnp}(-h)D^{(1)}_{mnp}-D^b_{1mn}=0$$

$$\sum_{p=1}^{6} G^{(1)}_{mnp}(-h)D^{(1)}_{mnp}-D^b_{2mn}=0$$

$$\sum_{p=1}^{6}[P^{(N)}_{mnp}(h)-\rho^t h^t\omega^2 H^{(N)}_{mnp}(h)]D^{(N)}_{mnp}=0$$

$$\sum_{p=1}^{6} F^{(N)}_{mnp}(h)D^{(N)}_{mnp}-D^t_{1mn}=0$$

$$\sum_{p=1}^{6} G^{(N)}_{mnp}(h)D^{(N)}_{mnp}-D^t_{2mn}=0. \quad (52)$$

Thus we have satisfied all the governing equations and boundary conditions, eqns (17)–(21). In summary, for fixed m and n, we need to solve the following system of sixteen equations obtained from eqns (45), (48), (49), (51) and (52)

$$\sum_{p=1}^{6} R^{(1)}_{mnp}(-h)D^{(1)}_{mnp}+h^b(\rho^b\omega^2-\bar{c}^b_{11}\alpha_m^2-c^b_{66}\beta_n^2)D^b_{1mn}-h^b(\bar{c}^b_{12}+c^b_{66})\alpha_m\beta_n D^b_{2mn}$$

$$=\frac{4\bar{e}^b_{31}\bar{V}^b}{ab\beta_n}[(-1)^m-1][(-1)^n-1]$$

$$\sum_{p=1}^{6} Q^{(1)}_{mnp}(-h)D^{(1)}_{mnp}-h^b(\bar{c}^b_{12}+c^b_{66})\alpha_m\beta_n D^b_{1mn}$$

$$+h^b(\rho^b\omega^2-c^b_{66}\alpha_m^2-\bar{c}^b_{11}\beta_n^2)D^b_{2mn}=\frac{4\bar{e}^b_{31}\bar{V}^b}{ab\alpha_m}[(-1)^m-1][(-1)^n-1]$$

$$\sum_{p=1}^{6} R_{mnp}^{(N)}(h)D_{mnp}^{(N)}+h^t(\bar{c}_{11}^t\alpha_m^2+c_{66}^t\beta_n^2-\rho^t\omega^2)D_{1mn}^t+h^t(\bar{c}_{12}^t+c_{66}^t)\alpha_m\beta_n D_{2mn}^t$$
$$= -\frac{4\bar{e}_{31}^t\bar{V}^t}{ab\beta_n}[(-1)^m-1][(-1)^n-1]$$

$$\sum_{p=1}^{6} Q_{mnp}^{(N)}(h)D_{mnp}^{(N)}+h^t(\bar{c}_{12}^t+c_{66}^t)\alpha_m\beta_n D_{1mn}^t+h^t(c_{66}^t\alpha_m^2+\bar{c}_{11}^t\beta_n^2-\rho^t\omega^2)D_{2mn}^t$$
$$= -\frac{4\bar{e}_{31}^t\bar{V}^t}{ab\alpha_m}[(-1)^m-1][(-1)^n-1]$$

$$\sum_{p=1}^{6}[P_{mnp}^{(1)}(-h)+\rho^b h^b\omega^2 H_{mnp}^{(1)}(-h)]D_{mnp}^{(1)} = 0$$

$$\sum_{p=1}^{6} F_{mnp}^{(1)}(-h)D_{mnp}^{(1)}-D_{1mn}^b = 0$$

$$\sum_{p=1}^{6} G_{mnp}^{(1)}(-h)D_{mnp}^{(1)}-D_{2mn}^b = 0 \qquad (53)$$

$$\sum_{p=1}^{6}[P_{mnp}^{(N)}(h)-\rho^t h^t\omega^2 H_{mnp}^{(N)}(h)]D_{mnp}^{(N)} = 0$$

$$\sum_{p=1}^{6} F_{mnp}^{(N)}(h)D_{mnp}^{(N)}-D_{1mn}^t = 0$$

$$\sum_{p=1}^{6} G_{mnp}^{(N)}(h)D_{mnp}^{(N)}-D_{2mn}^t = 0$$

$$\begin{bmatrix} D_{mn1}^{(1)} \\ D_{mn2}^{(1)} \\ D_{mn3}^{(1)} \\ D_{mn4}^{(1)} \\ D_{mn5}^{(1)} \\ D_{mn6}^{(1)} \end{bmatrix} = [T]^{(1)}[T]^{(2)}\dots[T]^{(N-1)} \begin{bmatrix} D_{mn1}^{(N)} \\ D_{mn2}^{(N)} \\ D_{mn3}^{(N)} \\ D_{mn4}^{(N)} \\ D_{mn5}^{(N)} \\ D_{mn6}^{(N)} \end{bmatrix}$$

for the sixteen unknowns $D_{mnp}^{(1)}$, $D_{mnp}^{(N)}$, D_{1mn}^b, D_{2mn}^b, D_{1mn}^t, and D_{2mn}^t. We note that the solution is nontrivial only when m and n are both odd, representing deformations symmetric about $x_1 = a/2$ and $x_2 = b/2$. The vanishing of the determinant of the coefficient matrix of eqn (53) determines the resonance frequencies for the free vibrations of the system.

3.4. *Plate with one actuator*

If there is only one actuator, for example, affixed to the top surface of the plate, the boundary conditions at the bottom surface of the plate should be the vanishing of $\tau_{31}^{(1)}, \tau_{32}^{(1)}$ and $\tau_{33}^{(1)}$. We then have the following fourteen equations

$$\sum_{p=1}^{6} R_{mnp}^{(1)}(-h)D_{mnp}^{(1)} = 0$$

$$\sum_{p=1}^{6} Q_{mnp}^{(1)}(-h)D_{mnp}^{(1)} = 0$$

$$\sum_{p=1}^{6} R_{mnp}^{(N)}(h)D_{mnp}^{(N)}+h^t(\bar{c}_{11}^t\alpha_m^2+c_{66}^t\beta_n^2-\rho^t\omega^2)D_{1mn}^t+h^t(\bar{c}_{12}^t+c_{66}^t)\alpha_m\beta_n D_{2mn}^t = -\frac{4\bar{e}_{31}^t\bar{V}^t}{ab\beta_n}[(-1)^m-1][(-1)^n-1]$$

$$\sum_{p=1}^{6} Q_{mnp}^{(N)}(h)D_{mnp}^{(N)}+h^t(\bar{c}_{12}^t+c_{66}^t)\alpha_m\beta_n D_{1mn}^t+h^t(c_{66}^t\alpha_m^2+\bar{c}_{11}^t\beta_n^2-\rho^t\omega^2)D_{2mn}^t = -\frac{4\bar{e}_{31}^t\bar{V}^t}{ab\alpha_m}[(-1)^m-1][(-1)^n-1]$$

$$\sum_{p=1}^{6} [P_{mnp}^{(1)}(-h)+\rho^b h^b\omega^2 H_{mnp}^{(1)}(-h)]D_{mnp}^{(1)} = 0$$

$$\sum_{p=1}^{6} [P_{mnp}^{(N)}(h)-\rho^t h^t\omega^2 H_{mnp}^{(N)}(h)]D_{mnp}^{(N)} = 0 \tag{54}$$

$$\sum_{p=1}^{6} F_{mnp}^{(N)}(h)D_{mnp}^{(N)}-D_{1mn}^t = 0$$

$$\sum_{p=1}^{6} G_{mnp}^{(N)}(h)D_{mnp}^{(N)}-D_{2mn}^t = 0$$

$$\begin{bmatrix} D_{mn1}^{(1)} \\ D_{mn2}^{(1)} \\ D_{mn3}^{(1)} \\ D_{mn4}^{(1)} \\ D_{mn5}^{(1)} \\ D_{mn6}^{(1)} \end{bmatrix} = [T]^{(1)}[T]^{(2)}\dots[T]^{(N-1)} \begin{bmatrix} D_{mn1}^{(1)} \\ D_{mn2}^{(N)} \\ D_{mn3}^{(N)} \\ D_{mn4}^{(N)} \\ D_{mn5}^{(N)} \\ D_{mn6}^{(N)} \end{bmatrix}$$

for the fourteen unknowns $D_{mnp}^{(1)}, D_{mnp}^{(N)}, D_{1mn}^t$, and D_{1mn}^t.

4. NUMERICAL RESULTS

As an example, we consider a graphite-epoxy plate with PZT-G1195 actuators affixed on the top and bottom surfaces. The material parameters for the graphite-epoxy are

$$E_{11} = 1500\,\text{GPa},\quad E_{22} = E_{33} = 9\,\text{GPa},\quad \nu_{12} = \nu_{23} = \nu_{13} = 0.3, \tag{55}$$
$$G_{12} = G_{31} = 7.1\,\text{GPa},\quad G_{23} = 2.5\,\text{GPa},\quad \rho = 1600\,\text{kg/m}^3,$$

and those for the PZT-G1195 are

$$e^b = e^t = \begin{bmatrix} 0 & 0 & -2.1 \\ 0 & 0 & -2.1 \\ 0 & 0 & 9.5 \\ 0 & 9.2 & 0 \\ 9.2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^T \frac{C}{m^2}$$

$$c^b = c^t = \begin{bmatrix} 148 & 76.2 & 74.2 & 0 & 0 & 0 \\ & 148 & 74.2 & 0 & 0 & 0 \\ & & 131 & 0 & 0 & 0 \\ \text{Symmetric} & & & 25.4 & 0 & 0 \\ & & & & 25.4 & 0 \\ & & & & & 35.9 \end{bmatrix}$$

$$\rho^b = \rho^t = 7500\,\mathrm{kg/m^3}. \tag{56}$$

For geometric dimensions, we choose $a = 8$ cm, $b = 8/\sqrt{2}$ cm, $2h = 0.2$ cm, $h^b = h^t = 0.02$ cm. We also choose $\bar{V}^b = -\bar{V}^t = 50$ V. We note that the analysis presented above is valid for a laminated elastic plate. However, there is only one lamina in the example considered to facilitate the interpretation of computed results.

The structure has a series of natural bending vibration frequencies which can be ordered as ω_{mn}, $m, n = 1, 2, 3, \ldots$. The free bending vibration modes corresponding to ω_{11}, ω_{31}, ω_{13}, $\omega_{33}, \ldots$ are symmetric about both $x_1 = a/2$ and $x_2 = b/2$ and those corresponding to ω_{12}, ω_{21}, $\omega_{22}, \ldots$ are antisymmetric about either $x_1 = a/2$ or $x_2 = b/2$, or both. Only fully symmetric modes can be excited in our problem because of the symmetry of the structure and the loading conditions. The natural frequencies of the structure can be roughly estimated from the results of the plate theory. When the inertia and rigidity of the actuators are neglected, we have (Jones, 1975)

$$\omega_{mn} \approx \frac{\pi^2}{\sqrt{2h\rho}}\left[D_{11}\left(\frac{m}{a}\right)^4 + 2(D_{12}+2D_{66})\left(\frac{m}{a}\right)^2\left(\frac{n}{b}\right)^2 + D_{22}\left(\frac{n}{b}\right)^4\right]^{1/2} \tag{57}$$

or

$$\Omega_{mn} = \omega_{mn}\bigg/\left[\pi^2\left(\frac{D_{11}}{2\rho h}\right)^{1/2}\frac{1}{a^2}\right] \approx \left[m^4 + 2\left(\frac{a}{b}\right)^2\frac{(D_{12}+2D_{66})}{D_{11}}m^2n^2 + \left(\frac{a}{b}\right)^4\left(\frac{D_{22}}{D_{11}}n^4\right)\right]^{1/2} \tag{58}$$

where $D_{\alpha\beta} = E_{\alpha\beta}(2h)^3/12(1-\nu^2)$, $\alpha, \beta = 1, 2, 6$ is the flexural rigidity and Ω_{mn} is the normalized natural frequency. From eqn (58) we obtain

$$\Omega_{11} \approx 1.3, \quad \Omega_{31} \approx 4.95, \quad \Omega_{13} \approx 9.23, \quad \Omega_{33} \approx 11.7\ldots \tag{59}$$

We plot $|U_3(a/2, b/2, 0)| = |u_3(a/2, b/2, 0)|hc^p_{31}h^b/a^2e^p_{31}\bar{V}$, the normalized deflection of the centroid of the plate, as a function of the normalized forcing frequency

$$\Omega = \omega\bigg/\left[\pi^2\left(\frac{D_{11}}{2\rho h}\right)^{1/2}\frac{1}{a^2}\right]$$

in Fig. 2. It is seen that $U_3(a/2, b/2, 0)$ becomes large at certain discrete values of Ω, which signifies the resonance phenomenon. Those values of Ω at which resonances occur should be in the sequence Ω_{11}, Ω_{31}, Ω_{13}, $\Omega_{33}, \ldots$. The values of Ω_{11}, Ω_{31}, Ω_{13}, and Ω_{33} shown in Fig. 2 are $\Omega_{11} \approx 1.612$, $\Omega_{31} \approx 7.685$, $\Omega_{13} \approx 8.365$ and $\Omega_{33} \approx 12.738$. These values differ noticeably from those listed in eqn (59). That this difference is caused by the presence of actuators was verified by reducing the elasticities, mass density and the thickness of the actuators by 10^4, 10^5 and 10^3, respectively. For this case, the computed resonance frequencies equalled 1.296, 4.826, 8.697 and 10.965, respectively which are close to those listed in eqn (59). In Fig. 2, only the locations of the peaks are important which determine the resonant frequencies of the system. The relative magnitudes of the peaks depend on how close the sampling points of Ω are to the exact values of Ω_{mn} when the curve is computed.

Normalized deflection of the middle surface $U_3(x_1, x_2, 0) = u_3(x_1, x_2, 0)hc^p_{31}h^b/a^2e^p_{31}\bar{V}$, normalized shear stresses at the interface between the top actuator and the plate $\bar{T}_{31}(x_1, x_2, h) = \tau_{31}(x_1, x_2, h)h^b/e^p_{31}\bar{V}$ and $\bar{T}_{32}(x_1, x_2, h) = \tau_{32}(x_1, x_2, h)h^b/e^p_{31}\bar{V}$ are plotted in Figs 3–5, respectively, for Ω near Ω_{11}. Similar results for Ω near Ω_{31} are plotted in Figs 6–8, for Ω near Ω_{13} in Figs 9–11, and for Ω near Ω_{33} in Figs 12–14.

Under a particular forcing frequency Ω, all of the fully symmetric free vibration modes may be excited. But when Ω is close to a particular resonant frequency, the mode

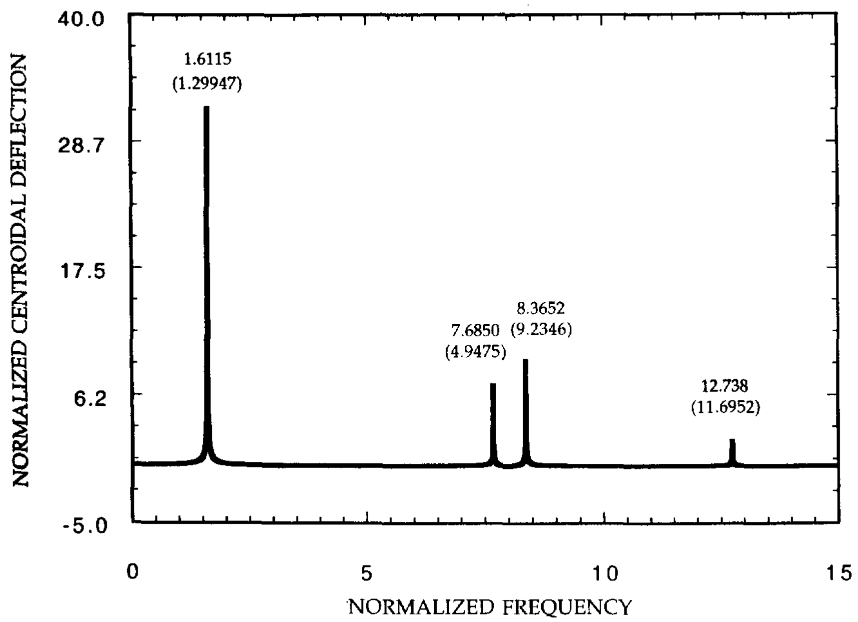

Fig. 2. The normalized deflection of the centroid $|U_3(a/2, b/2, 0)| = |u_3(a/2, b/2, 0)|hc^p_{31}h^b/a^2e^p_{31}\bar{V}$ as a function of the nondimensional forcing frequency $\Omega = \omega/(\pi^2(D/2\rho h)^{1/2}\ 1/a^2)$.

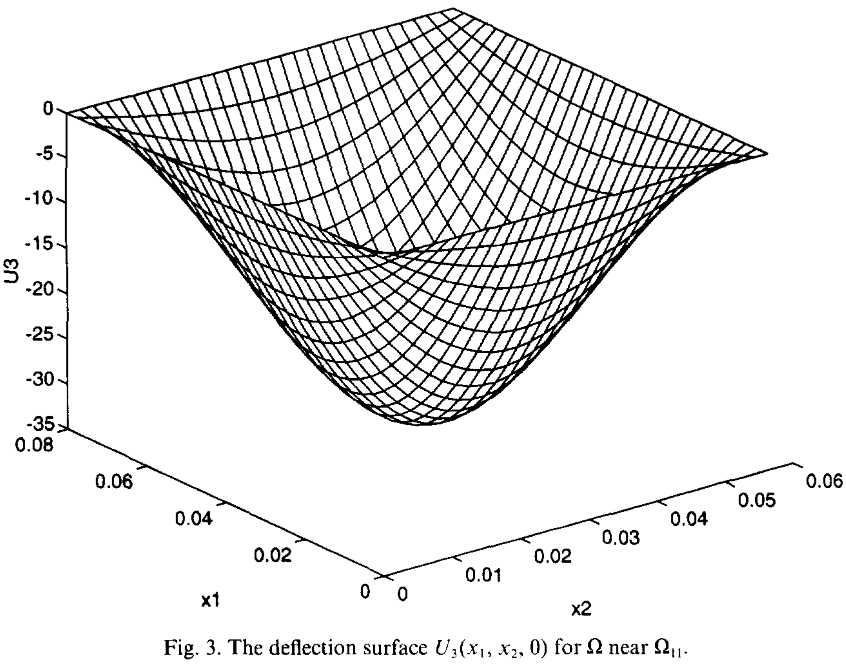

Fig. 3. The deflection surface $U_3(x_1, x_2, 0)$ for Ω near Ω_{11}.

corresponding to that resonant frequency has a dominant contribution to the deflection of the plate. The Fourier series converges very fast. When Ω is near Ω_{11}, Ω_{31}, Ω_{13}, or Ω_{33}, at most 20 terms are needed for $u_3(a/2, b/2, 0)$ to have four significant digits. When Ω is higher, higher order modes also become important hence more terms are needed in the series. For all of the results presented herein, *eight hundred terms* in the Fourier series are summed to ensure sufficient accuracy.

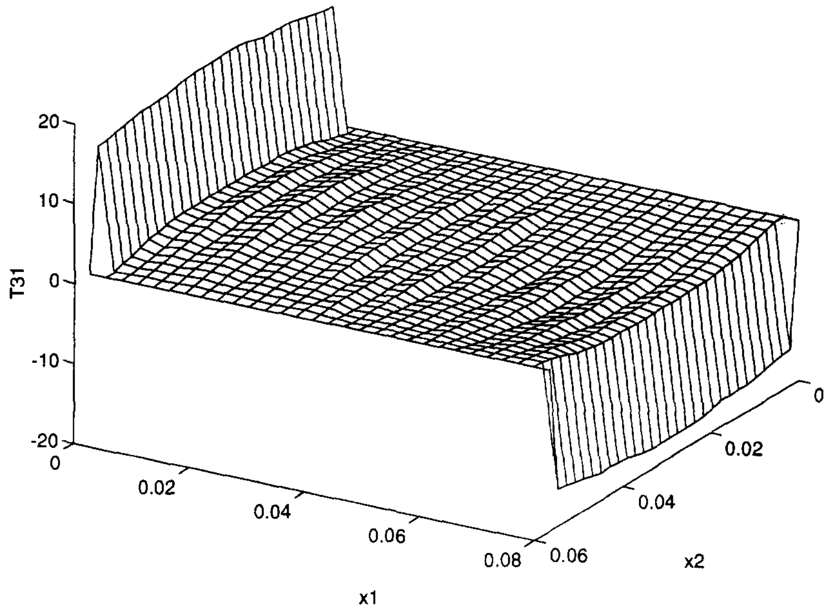

Fig. 4. The normalized shear stress $T_{31}(x_1, x_2, h) = \tau_{31}(x_1, x_2, h)h^b/e^p_{31}\bar{V}$ under the top actuator for Ω near Ω_{11}.

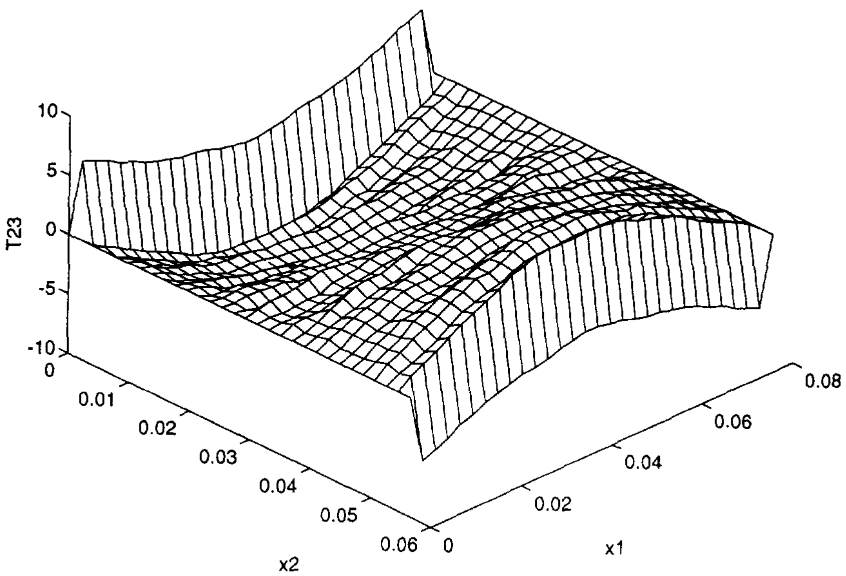

Fig. 5. The normalized shear stress $T_{32}(x_1, x_2, h) = \tau_{32}(x_1, x_2, h)h^b/e^p_{31}\bar{V}$ under the top actuator for Ω near Ω_{11}.

It can be seen that for vibrations in modes 11, 31, 13 and 33, the normalized shear stress $\bar{T}_{31}(x_1, x_2, h)$ or $\bar{T}_{32}(x_1, x_2, h)$ at points on the interface between the top actuator and the plate surface is mainly concentrated in a narrow region near the edges. This is similar to the results for beams computed by Hanagud and Kulkarni (1992) by the two-dimensional finite element method and by Zhou and Tiersten (1994) and Yang *et al.* (1994), who used the two-dimensional elasticity theory. In the approximate plate theory, these shear stresses have delta function distributions (Zhou and Tiersten, 1994). The finite element method

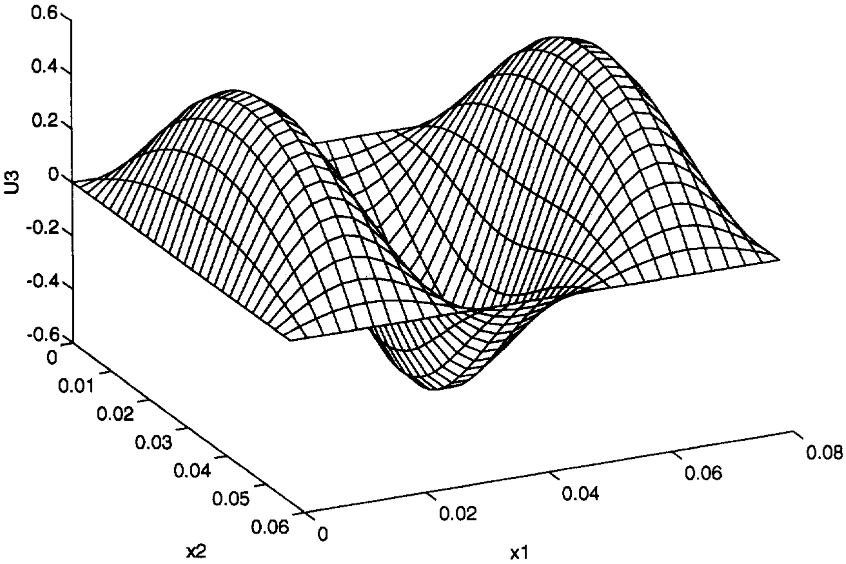

Fig. 6. The deflection surface $U_3(x_1, x_2, 0)$ for Ω near Ω_{31}.

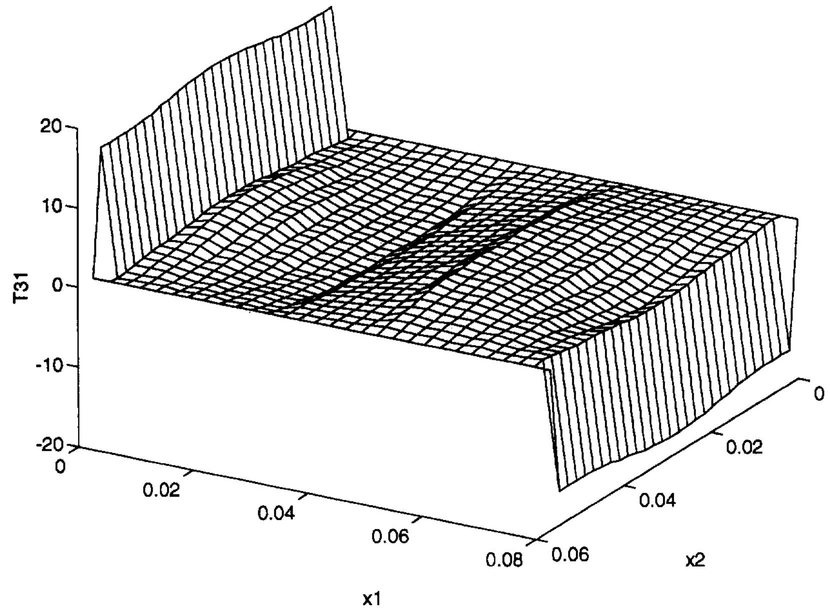

Fig. 7. The normalized shear stress $T_{31}(x_1, x_2, h) = \tau_{31}(x_1, x_2, h)h^b/e^p_{31}\bar{V}$ under the top actuator for Ω near Ω_{31}.

usually predicts a more gradual change of the shear stress distributions which is determined by the element size and the interpolation functions used (Hanagud and Kulkarni, 1992). The results by Fourier series sometimes have slight oscillations in the shear stress distribution near the edges (Zhou and Tiersten, 1994), which looks like the Gibbs phenomenon of a Fourier series near a jump discontinuity.

5. CONCLUSIONS

We have presented a Fourier series analysis of the vibrations of an elastic rectangular plate forced by piezoelectric actuators under time harmonic electric voltage. The solution

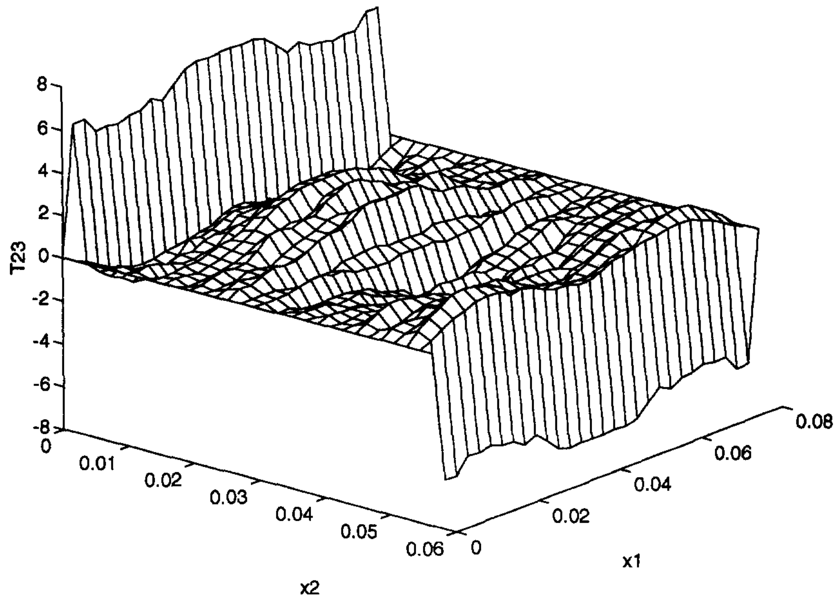

Fig. 8. The normalized shear stress $T_{32}(x_1, x_2, h) = \tau_{32}(x_1, x_2, h)h^b/e^p_{31}\bar{V}$ under the top actuator for Ω near Ω_{31}.

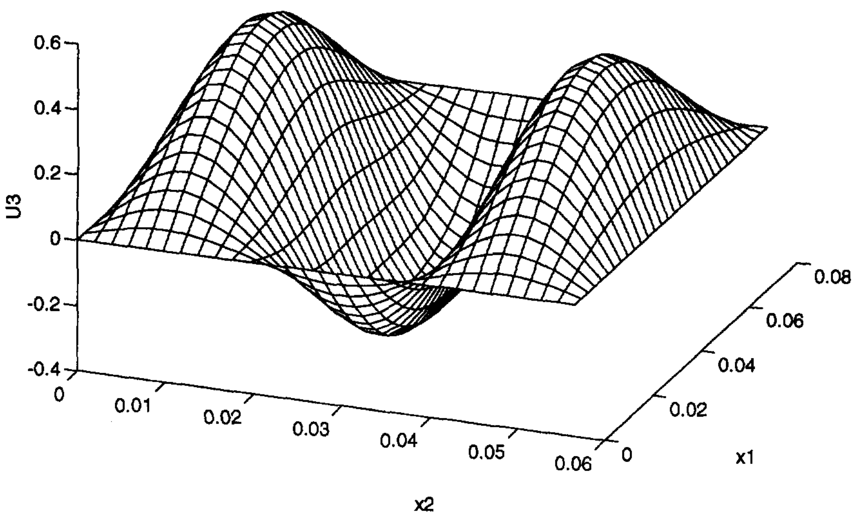

Fig. 9. The deflection surface $U_3(x_1, x_2, 0)$ for Ω near Ω_{13}.

is exact within the linear theory of elasticity. The Fourier series converges rapidly in the numerical example studied. For the graphite-epoxy plate with actuators attached to its top and bottom surfaces, it is shown that the normalized shear stress at the interface between the plate and an actuator is essentially zero except in very small regions near the edges for vibrations of the plate in modes 11, 13, 31 and 33. The computed natural frequencies for the orthotropic plate are found to be close to those estimated from the plate theory. However, PZT attached layers to the top and bottom surfaces of the plate shift noticeably the natural frequencies of the plate.

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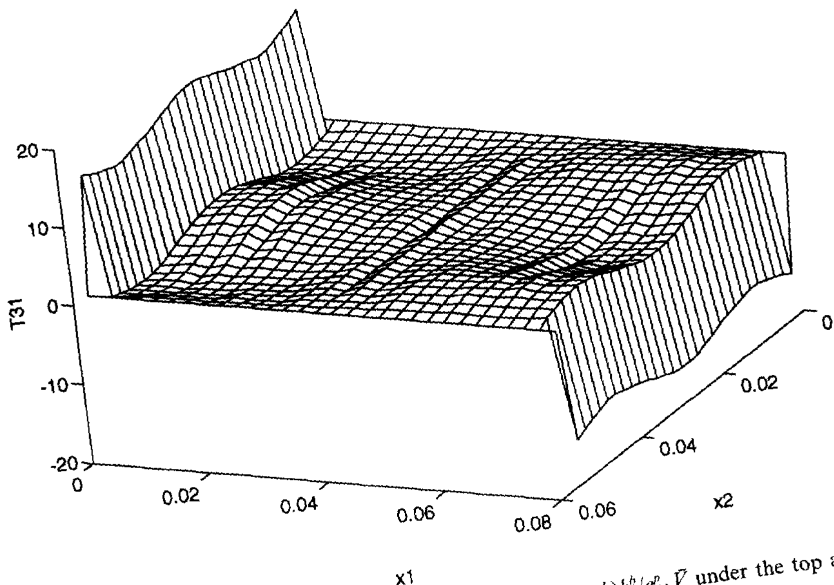

Fig. 10. The normalized shear stress $T_{31}(x_1, x_2, h) = \tau_{31}(x_1, x_2, h)h^b/e^p_{31}\bar{V}$ under the top actuator for Ω near Ω_{13}.

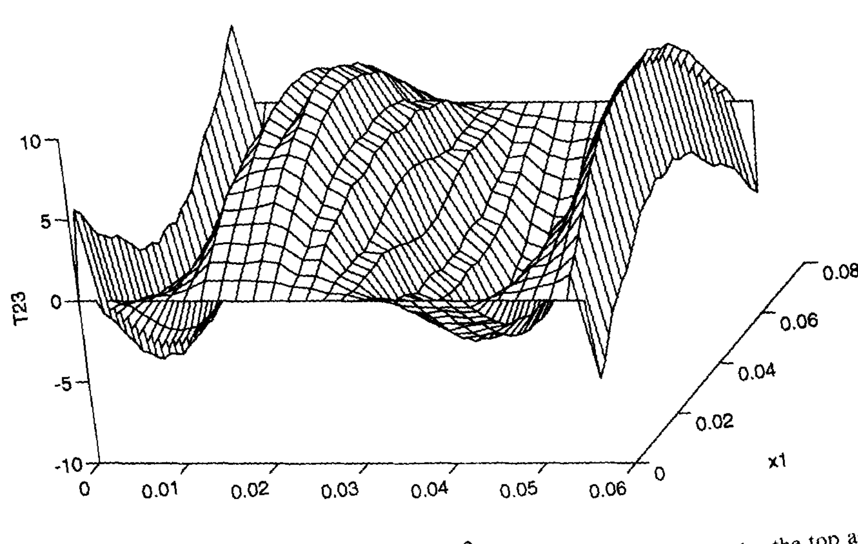

Fig. 11. The normalized shear stress $T_{32}(x_1, x_2, h) = \tau_{32}(x_1, x_2, h)h^b/e^p_{31}\bar{V}$ under the top actuator for Ω near Ω_{13}.

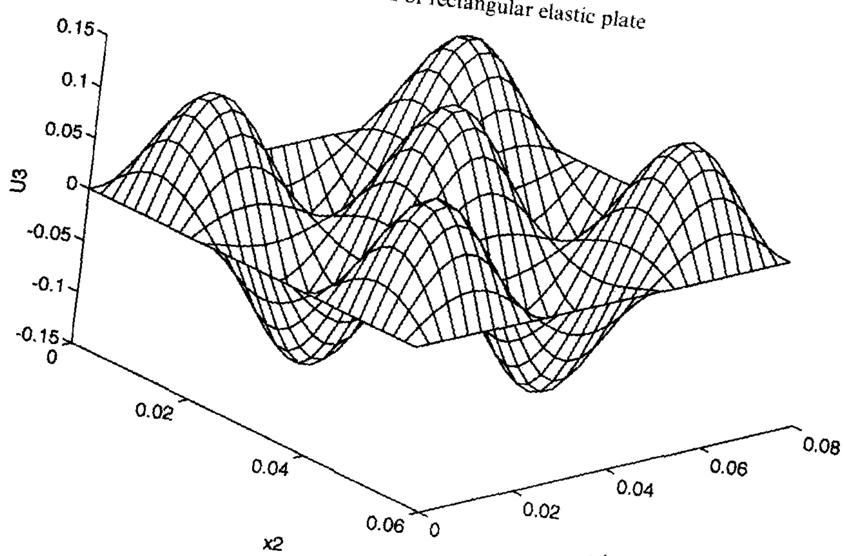

Fig. 12. The deflection surface $U_3(x_1, x_2, 0)$ for Ω near Ω_{33}.

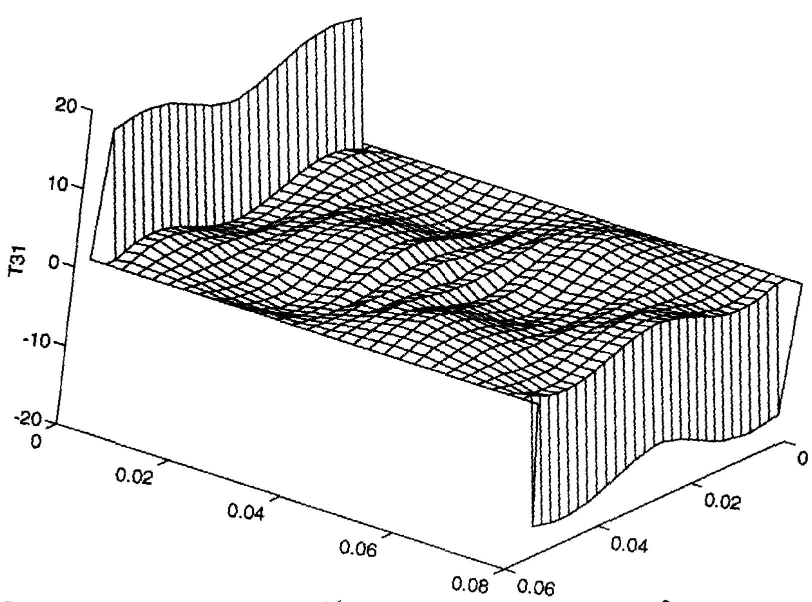

Fig. 13. The normalized shear stress $T_{31}(x_1, x_2, h) = \tau_{31}(x_1, x_2, h)h^p/e_{31}^p\bar{V}$ under the top actuator for Ω near Ω_{33}.

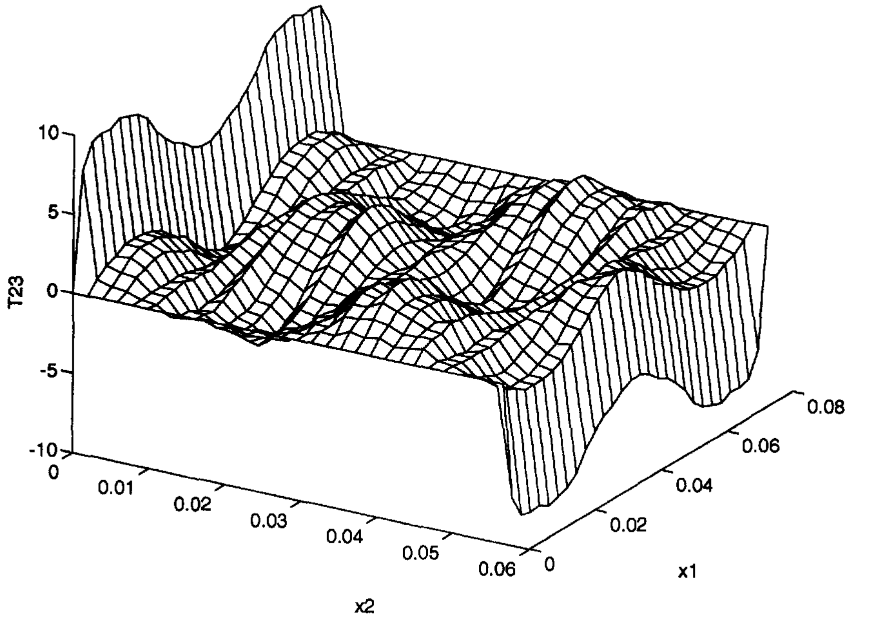

Fig. 14. The normalized shear stress $T_{32}(x_1, x_2, h) = \tau_{32}(x_1, x_2, h)h^b/e^p_{31}\bar{V}$ under the top actuator for Ω near Ω_{33}.

REFERENCES

Hanagud, S. and Kulkarni, G. (1992). Coupled piezoceramic-elastic structures with finite deformations. *Developments in Theoretical and Applied Mechanics, Proc. 16th Southeastern Conference on Theoretical and Applied Mechanics*, **XVI**, pp. II.20.22–30. The University of Tennessee Space Institute.

Jones, R. M. (1975). *Mechanics of Composite Materials*. Scripta, Washington, DC.

Mitchell, J. A. and Reddy, J. N. (1995). A refined hybrid plate theory for composite laminates with piezoelectric laminae. *Int. J. Solids Structures* **32**, 2345–2368.

Srinivas, S., Rao, C. V. J. and Rao, A. K. (1970). An exact analysis for vibration of simply-supported homogeneous and laminated thick rectangular plates. *J. Sound Vib.* **12**, 257–269.

Tang, Y. Y. and Xu, K. (1995). Dynamic analysis of a piezothermoelastic laminated plate. *J. Thermal Stresses* **18**, 87–104.

Tauchert, T. R. (1992). Piezothermoelastic behavior of a laminated plate. *J. Thermal Stresses* **15**, 25–37.

Tiersten, H. F. (1993). Electroelastic equations for electroded thin plates subject to large driving voltages. *J. Appl. Phys.* **74**, 3389–3393.

Wang, Z. K., (1987). *Mathematical Formulas*, pp. 16–17. Chongqing Press, China [in Chinese].

Wittrick, W. H. (1987). Analytical, three-dimensional elasticity solutions to some plate problems, and some observations on Mindlin's plate theory. *Int. J. Solids Structures* **23**, 441–464.

Yang, J. S., Batra, R. C. and Liang, X. Q. (1994). The cylindrical bending vibration of a laminated elastic plate due to piezoelectric actuators. *Smart Mater. Structures* **3**, 485–493.

Zauderer, E. (1983). *Partial Differential Equations of Applied Mathematics*, p. 167. Wiley, New York.

Zhou, Y. S. and Tiersten, H. F. (1994). An elastic analysis of laminated composite plates in cylindrical bending due to piezoelectric actuators. *Smart Mater. Structures* **3**, 255–265.